# Logicism and the Sense–Denotation Distinction

Mark Eli Kalderon

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## §1 Introduction

Unless you are a Frege scholar, or a philosopher of mathematics, if you are familiar at all with Frege's work, you are most likely familiar with his groundbreaking work in the philosophy of language. You might know that Frege was a mathematician who sought to establish the covertly logical subject matter of arithmetic, a project whose demands drove Frege to his logical investigations and reflections on language. But most likely the connection between Frege's mathematical research and his philosophy of language remains elusive for you.

We have only in the last two decades entered what is surely the golden age of Frege scholarship and are only now beginning to reap the full harvest of Frege's thought. However, if you turned to the work of Frege scholars for a clear account of the relationship betweeen Frege's mathematics and his philosophy of language you will surely be disappointed. This is surprising—once seen, the connection between Frege's logicism, on the one hand, and his distinction between sense and denotation, on the other, is both easily stated and easily comprehended.

Why do we lack a clear account of the relationship between Frege's logicism and the distinction between sense and denotation?

I believe this is due, in part, to difficulties we all face in reading Frege. Frege is not difficult to read in the way that Kant can be. Though Frege's writings are particularly lucid especially given the historical context in which they were produced, there are difficulties peculiar to reading Frege. The two main difficulties in reading Frege pull in opposite directions. On the one hand, many of

Frege's lessons have been assimilated to such an extent that we have a tendency to forget the fundamental insights that lie behind them. In reading Frege, we must relearn what, in a sense, we already know (however dimly or inexplicitly). On the other hand, because so many of Frege's lessons are familiar, we are apt to miss what is truly idiosyncratic in Frege. We are apt, that is, to make Frege more of a contemporary than he actually is.

It is this latter difficulty, especially with respect to understanding the puzzle that motivates the distinction between sense and denotation, that is the source of the difficulty. Frege's eponymous puzzle is indeed Frege's, though he understands it differently than we do. This is partly a difference in emphasis and partly a difference in substance. What we understand as Frege's puzzle is equivalent, modulo a substantive and controversial thesis of Fregean grammar, to another puzzle that is Frege's real concern. Once we have this latter puzzle in view, the connection between Frege's logicism and the distinction between sense and denotation will be clear.

The account connecting Frege's logicism and the sense–denotation distinction spans from the unfinished business of the *Foundations* through "Function and Concept" to "On Sense and Denotation". However, to motivate that account, I want to begin with "On Sense and Denotation" and the mismatch between the structure of that article and our contemporary understanding of Frege's eponymous puzzle.

## §2 The Structure of "Sense and Denotation"

Frege's puzzle *as we understand it*, might also be described as *The Puzzle About Substitution*: How can two sentences that differ only in the substitution of codesignative names differ in cognitive value?

The Puzzle About Substitution can be spelled out as follows: Suppose, naïvely, that there is nothing more to the meaning of a name than the object that it denotes. "Hesperus" and "Phosphorus" are codesignative—they each denote the same object, the planet Venus. It follows that "Hesperus" and "Phosphorus" must mean the same. If "Hesperus" and "Phosphorus" mean the same, then one can substitute one for the other in any sentence without changing the meaning of that sentence. "Hesperus" is a meaningful constituent of the sentence, "Hesperus is visible in the evening". Since "Phosphorus" means the same as "Hesperus" one can substitute "Phosphorus" in for "Hesperus" without changing the meaning of that sentence. But "Phosphorus is visible in the evening" does not mean the same as "Hesperus is visible in the evening"—they differ in *cognitive value*. A competent speaker who understands these sentences can regard one as true without regarding the other as true—perhaps he regards it as false, or at the very least, it is an open question for him. But if "Hesperus is visible in the evening" and "Phosphorus is visible in the evening" differ in meaning, then since they differ only in the substitution of "Phosphorus" in for "Hesperus", "Hesperus" and "Phosphorus" do not themselves mean the same. We have a puzzle, an inconsistent set of claims, and the Fregean recommends that we best avoid it by abandoning the naïve claim that the meaning of a name is the object that it denotes.

The Puzzle About Substitution *can* be found in "On Sense and Denotation", but it is not its sole or even central concern. A failure to appreciate this could only be due to a myopic concern with the opening passages of "On Sense and Denotation".

If one takes the central concern of "On Sense and Denotation" to be The Puzzle About Substitution, then the structure of that essay and its subject matter ought to be puzzling.

The first thing to notice is that the discussion of identity and identity sentences that inagurates that essay is held to be the *locus classicus* of The Puzzle About Substitution and yet is clearly meant simply to introduce the sense–denotation distinction. The specific topic of these remarks, the meaning and subject matter of identity sentences, is dealt with swiftly and never again mentioned until the last pargraph and then only perfunctorily. The introductory character of this passage is further confirmed by the fact that an argument for the sense–denotation distinction is more fully developed later in the essay. (For example, the important role of compositionality is made explicit in that argument in the way that it is not in the inagural discussion of identity sentences). But, as we will see, both its form and its context indicate a concern with something other than The Puzzle About Substitution.

The introductory character of the inagural discussion of identity sentences is a matter of interpretation, and some may put little weight on it. However, in the next thirteen paragraphs the distinction between sign, sense, and denotation is explained further: Frege has clearly moved on from *arguing* for the distinction—if indeed he has argued for it at all—to *explaining* it. Let's briefly review Frege's explanation.

Paragraph two explicitly introduces the notion of sense "wherein the mode of presentation is contained". Pargraph three clarifies that the notion of sign at issue is that of a proper name. Paragraph four emphasizes the partial character of modes of presentations: "The sense of a proper name ... serves to illuminate only a single aspect of the thing meant." The partial character of modes of presentations will be echoed in the astronomical analogy discussed in paragraph ten.

Paragraph five discusses the relationship between sign, sense, and denotation. Different signs can have the same sense. Ideally, every sign should have a definite sense, but this is not true of natural languages (though it is enough that natural language expressions have a definite sense in context). While every grammatical expression should have *a* sense, not every grammatical expression has a denotation. (Thus, for example, "the least rapidly converging series" is a grammatical expression that has sense but no denotation.)

Paragraphs six and seven introduce an important theme that will be explored in detail later in the essay with respect to a variety of constructions. Paragraph six observes that words lack their ordinary denotation in quotation contexts—they are instead "signs for signs". Paragraph seven develops this further with respect to indirect discourse. "A" as it occurs in the complex expression "The sense of the expression 'A'" does not have its customary denotation but denotes, instead, its customary sense. Frege thus distinguishes the *customary* and *indirect denotation* of a word and the *customary* and *indirect sense* of a word. In indirect discourse, words have as their indirect denotation their customary sense, itself presented under a mode of presentation, their indirect sense.

Paragraphs eight through ten distinguish the sense and denotation of a name from ideas. Paragraph eight distinguishes any idea associated with a name from its denotation. Different people have different ideas of a thing, and these ideas are subjective, and yet their use of a name has the same denotation (very often itself something objective). Paragraph nine distinguishes any idea associated with a name from its sense. While in natural language different speakers may associate different senses with the same name, senses, unlike ideas, are objective–they are available in principle to different speakers in the way that ideas are in principle not. The discussion of paragraphs eight and nine are summmed up in the central astronomical analogy of paragraph ten:

> Somebody observes the Moon through a telescope. I compare the Moon itself to the meaning; it is the object of observation, mediated by the real image projected by the object glass in the interior of the telescope, and by the retinal image of the observer. The former I compare to the sense, the latter is like the idea or experience.

The optical image provides a partial perspective on the Moon—it is "one-sided and dependent upon the standpoint of observation"—and yet is objective insofar as it is a perspecitive that can, in principle, be taken up by others. The retinal image, in contrast, belongs to one person and one person alone.

Paragraph eleven announces the end of the discussion of ideas, and paragraph twelve regiments our semantic terminology—a name *expresses* its sense and *denotes* its denotation. Paragraph thirteen argues in response to the idealist or the skeptic that in using "The Moon" we do not intend to speak of our idea of the Moon but merely *presuppose* that it has a denotation. Our use of a name may presuppose that it has a denotation, but that is not to assume that that presupposition is justified in any particular case and so the complaint of the idealist or skeptic has no purchase.

So far so familiar. But paragraph fourteen marks a decisive shift in topic that will occupy Frege for the rest of the essay. Up until this point, Frege has been discussing the sense and denotation of proper names. Frege now considers the sense and denotation of *sentences*. Over the next seven paragraphs Frege defends, in a preliminary fashion, four distnguishable theses:

1. Sentences have denotations
2. The denotations of sentences are truth values
3. Names and sentences belong to a unified logical category—in Frege's mature vocabulary, they are each *complete* expressions
4. Truth values are objects.

In paragraph nineteen, Frege writes:

> If the supposition that the *Bedeutung* of a sentence is its truth-value is correct, the latter must remain unchanged when a part of the sentence is replaced by an expression with the same *Bedeutung*. And this is in fact the case. Leibniz gives the definition: "*Eadem sunt, quae sibi mutuo substitui possunt, salva veritate*". [They are the same which may be substituted one for another while preserving truth.]

In paragraph twenty one, Frege considers a related test of the hypothesis that the denotation of a sentences is its truth-value: "We have found that the truth-value of a sentence remains unchanged when an expression in it is replaced by another with the same *Bedeutung*: but we have yet considered the case in which the expression to be replaced is itself a sentence." This leads Frege to a discussion of subordinate clauses, a discussion that will occupy him for the remainder and indeed the bulk of the essay.

Notice the Puzzle About Substitution can be developed independently of the assumption that the denotation of a sentence is its truth-value and, indeed, independently of the assumption that sentences have denotations at all (at least if denotations are understood as semantic values in common between proper names and sentences). When we first explained The Puzzle About Substitution we did so without making either of these assumptions. Contemporary discussions of The Puzzle About Substitution very often consider subordinate clauses—for example, we get discussions of propositional attitude constructions. (Though these constructions are unnecessary to state The Puzzle About

Substitution—when we first stated the puzzle we did so with simple as opposed to complex sentences.). However, the concern for subordinate clauses in contemporary discussions of The Puzzle About Substitution are crucially different from Frege's. The contemporary concern is with the substitution of codesignative names embedded within subordinate clauses; Frege's avowed concern is with the substitution of codesignative sentences embedded within subordinate clauses. So, the best part of "Sense and Denotation" is not devoted to The Puzzle About Substitution at all but is, rather, devoted to defending the thesis that the denotation of a sentence is its truth-value.

Given modern expectations this ought to seem very odd. This is not the essay that a modern-day Fregean would write if he were to do so *ab initio*. A modern-day Fregean, a philosopher moved by the The Puzzle About Substitution to posit the distinction between sense and denotation, would not devote the best part of an essay introducing that puzzle and its consequences to what is, in effect, an orthogonal issue. So what is this discussion of truth doing here? And why does Frege evidently believe it to be so important?

## §3 The Unfinished Business of *The Foundations*

In *The Foundations of Arithmetic*, Frege argues for *logicism*$_\llcorner$, the view that the truths of arithmetic can be derived from definitions and the laws of logic. Frege distinguishes mathematical (§§1 & 2) and philosophical (§§3 & 4) motivations for his foundational work in arithmetic.

The mathematical project is to restore the "old Euclidean standards of rigor" by providing proofs of arithmetical claims that have hitherto been regarded as self-evident—proofs which satisfy two conditions:

1. every assumption is explicitly stated; and

2. every inferential transition is in accord with an acknowledged rule

If the target mathematical claims are indeed self-evident, then what's the point? Frege cites four related reasons for his project of proving the obvious:

1. The renewed impulse of rigor in geometry and analysis has already shown itself mathematically fruitful by revealing the "limits of validity" of certain important theorems.

2. By making explicit the inferential principles that implicitly guide our judgments we may arrive at "general methods" of forming concepts and proving theorems that may help us resolve open mathematical questions.

3. By reducing the number of judgments that are accepted without proof we achieve an economy that is valuable in its own right.

4. Even if the truth of a thought is beyond doubt and the limits of its validity is perfectly clear, it is still a mathematical advance to prove it—in this way we reveal the pattern of logical dependencies among thoughts thereby clarifying the content of our mathematical theories.

If one is going to dedicate one's lifework to proving self-evident arithmetical claims reasons one and two may seem a little thin. There is no guarantee at the outset that the limits of validity of important theorems will be established or that any general methods of concept formation or mathematical proof will be discovered. The economy put forth by reason three is merely an aesthetic virtue of which Frege makes little theoretical use. Frege's real motive lies with the thought that discovering proofs where proof is available is a genuine mathematical advance.

"Philosophical motives too ... prompted [Frege] to enquiries of this kind". What's the epistemological status of arithmetical truths? Are arithmetical truths analytic a priori, synthetic a priori, or synthetic a posteriori? According to Frege, the distinction between a priori and a posteriori judgments concerns the grounds or type of justification for such judgments. A posteriori judgments are judgments that are grounded in or justified by the empirical intuition. However, we may be justified in accepting certain judgments independently of the contingent course of empirical intuition. Similarly, analytic judgments, as Frege conceives of them, are grounded in the content of the judgment. We only need to apprehend the constituent concepts of the judgment that all bachelors are unmarried males to recognize its truth. Synthetic judgments, on the other hand, cannot be justified on the basis of their content alone but must be justified at least partly on the basis of inutition whether empirical or pure.

In the *First Critique*, Kant provides two distinct characterizations of analyticity:

- A judgment is *analytic* just in case the subject concept contains the predicate concept.
- A judgment is *analytic* just in case its denial is self-contradictory.

Three observations are relevant:

1. On the first characterization, our understanding of analyticity is only as clear as our understanding of conceptual containment. But talk of containment is metaphorical at best. Unless the literal content of containment is made explicit, we will lack a clear understanding of analyticity.

2. On the second characterization, our understanding of analyticity is only as good as the underlying logic. Kant however still accepts the term logic inherited from Aristotle.

3. The two characterizations are nonequivalent. Kant's characterization of analyticity in terms of conceptual containment presupposes an oversimplified view of judgment—this account only applies to universal affirmative judgments, i.e., judgments of the form: All *A*s are *B*s. But many judgments, e.g., $23 > 17$, do not take this form. Not only do singular judgments not take this form but neither do existential judgments: there is a square root of 16. The notion of analytic judgments as judgments whose denials are self-contradictory provides a broader notion of analyticity. The notion of logical consistency is not restricted to universal affirmative judgments, and so the second characterization has a broader range of applicability. So not only are different concepts deployed in Kant's two characterizations, but they are not even extensionally equivalent.

Frege's reconstrual of Kant's notion of analyticity at once resolves these difficulties and reconciles the distinct characterizations. In this regard, Frege's hermeneutic achievement is of Talmudic proportions:

> A truth is *analytic* just in case it can be transformed into a logical truth by the substitution of synonym for synonym.

Where for Frege, a logical truth is a truth that can be proved from the laws of logic alone.

So consider:

> All bachelors are unmarried males.

If the expressions "bachelor" and "unmarried male" are indeed synonyms (if they share the same conceptual content), then by compositionality we may substitute the latter in for the former without altering the thought expressed and thus arrive at a truth of logic:

> All unmarried males are unmarried males.

The denial of a logical truth is self-contradictory, and so Frege's characterization is faithful to the spirit of Kant's logical characterization of analyticity. Moreover, that logical truths are arrived at by the substitution of synonym for synonym explicates Kant's talk of conceptual containment.

Not only has Frege reconstrued the Kantian distinction between the analytic and the synthetic, but what has been less well appreciated is that Frege has also

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reconstrued the distinction between a priori and a posteriori judgment. For Kant, a priority was an *epistemological* notion. What counts as a priori was in the first instance a cognitive act, a bit of knowledge. And to call a cognition a priori is to say that it somehow prior to or independent of experience (for Kant, empirical intuition). That is, at least ever since Kant, philosophers have recognized a fundamental epistemological distinction between that knowledge which is based on experience or empirical intuition or observation and that which is somehow not.

Given this, Frege's characterization of the a priori ought to strike us as peculiar. For it contains no reference whatsoever to experience, empirical intuition, observation or any other experience-theoretic category. For Frege:

> A truth $S$ is *a priori* just in case there exists a proof of $S$ that does not depend on any basic facts about particular objects, that is, just in case there exists at least one proof of $S$ that involves only general truths as premises.

Frege seems to have provided a logical characterization of what has been previously construed as an epistemological notion. Whereas tradition speaks of experience and observation—modes or sources of knowledge and belief, Frege speaks only of general laws and facts about particulars. How then can Frege's answer to the question of the a priority of mathematical truths as Frege understands the notion, even count as an answer to the traditional questions about the epistemological status of mathematical truths, the questions that Plato, Leibniz, Mill and above all Kant were at pains to answer in their remarks about mathematical practice? Hasn't Frege just changed the subject here?

That's a good question. I won't answer it here. I will only observe that Frege shares a conception of the a priori with a pre-modern tradition whose traces can be found in Leibniz's writings and that probably this conception is bequeathed to Frege from Leibniz.

What's the epistemological status of arithmetical truths? Given Frege's refashioning of the Kantian categories, an adequate answer to this philosophical question will involve an answer to Frege's mathematical question—to what extent are arithmetical truths provable on the basis of definitions and logic alone? Thus the philosophical questions of the *Foundations* depend on the answers to mathematical questions. If the truths of arithmetic are provable on the basis of definitions and logic alone, then they will be analytic (since they will be transformable into logical truths via the substitution of synonyms) and a priori (since they will be provable on the basis of general laws without the supplemenation of intuition of any particular).

In the *Foundations*, Frege seeks to make the claim that arithmetic is provable on the basis of definitions and logic alone "merely probable". Frege advances three sorts of considerations:

1. A single positive argument, the *Generality Argument*, according to which arithmetic applies to not only the actual, not only the intuitable, but to everything thinkable which is the broadest domain of all and is, at the very least, coextensive with the proper domain of logic.
2. The refutation of extant alternatives in the middle of the book, e.g., his arguments against Kant, Mill, etc.
3. The definitions and proof sketches in §§55–87.

Frege will only take himself to have established the analyticity of arithmetic once the definitions and proof sketches of §§55–87 are formally executed in the *Basic Laws of Arithmetic*.

Frege's logicist reduction of arithmetic requires:

- that the truths of arithmetic be defined in purely logical terms, and
- that the truths of arithmetic be derivable from the basic laws of logic together with these definitions.

Frege defines the number belonging to the concept $F$ as the extension of the concept equinumerous to the concept F: $NxFx = \{x : x$ is equinumerous to the concept $F\}$

Equinumerosity sounds dangerously like *the same number as* so is this definition really noncircular in the way required? One important insight that Frege attributes to Hume is the role of one-one correlation in defining equinumerosity:

> When two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal.

Let's consider this step by step. Begin with the notion of *correlation*. It is a *relation* that correlates items in the extension of two concepts. Specifically the relation $\Phi$ *correlates* concepts $F$ and $G$ just in case for everything that falls under the concept $F$ there is something that falls under $G$ that $\Phi$ relates and for everything that falls under the concept $G$ there is something that falls under the concept $F$ that $\Phi$ relates.

Suppose that $\Phi$ correlates concepts $F$ and $G$. This correlation is *one-one* just in case it satisfies a further condition. Specifically, $\Phi$ correlates an item that falls under the concept F with one and only one thing that falls under the concept G, and $\Phi$ correlates an item that falls under the concept G with one and only one thing that falls under the concept F.

Hume's insight is that we can now define equinumerosity in terms of one-one correlation:

The concept $F$ is *equinumerous* to the concept $G$, $F \approx G$, $\equiv_{def}$ $\exists\Phi\Phi$ one-one correlates the concepts $F$ and $G$

Notice we have been able to define equinumerosity in purely logical terms without appealing to the concept the same number as. So it would seem that Frege's definition of number is noncircular in the manner required.

Given Frege's definition, further definitions are possible. Thus $n$ is a number iff $\exists F(n = NxFx)$. Moreover:

- $0 = Nx(x \neq x)$
- $1 = Nx(x = 0)$
- $2 = Nx(x = 0 \lor x = 1)$
- $n$ follows in the series of natural numbers directly after $m$, $s(m) = n$, iff $\exists F \exists x[Fx \land NyFy = n \land Nz(Fz \land z \neq x) = m]$

In terms of these definitions, Frege goes on to prove a number of laws of arithemtic and other important claims (such as the infinity of the natural numbers) on the basis of logic alone.

So, in *The Foundations* Frege proposes a definition of number as a kind of concept-extension and outlined how arithmetical truths can be derived from the general laws of logic in terms of this definition. But before the proof can be formally executed in the *Basic Laws of Arithemtic*, clarification is needed of the notions of concept and concept-extension. In the *Begriffsschrift*, Frege suggested that a functional analysis of concepts is required in order to explicate their logical function. Unfortunately, while relatively clear about the nature of the argument of such functions, that is, the denotations of non-empty singular terms, Frege was less than explicit about the *values* of such functions. Until Frege has specified the nature of the values of those functions with which concepts are identified, we have as yet no clear conception of Fregean concepts nor what he takes their extensions to be. It also follows, then, that we have no clear conception of what numbers are if they are to be, as Frege urges, a particular class of extensions, nor do we have a clear conception of the logical principles governing concept-extensions. It is to this project, clarifying the operative notions of concept and concept-extension, that Frege turns in his writings of his middle period. Finding a class of values for concepts will dictate the structure of his mature philosophy of language and logic. The connection between Frege's philosophy of language and philosophy of mathematics is precisely the specification of the fugitive values of those functions with which concepts are identified. Frege's decision, in conjunction with his commitment to the compositionality principle, will dictate his views about truth, his distinction between sense and reference, his identification of the content of assertions with thoughts, and so on. And make a decision he must, for without an explicit statement of what extensions are and of the logical principles that govern them, he cannot complete the system of proofs outlined in *The Foundations*. Thus Frege's essay, “Function and Concept,” is of first importance, perhaps even more important than “On Sense and Denotation.” It is there that he makes his crucial decision and thereby laid a foundation for his philosophy of language.

## §4 “Function and Concept” and The Transition of to the Middle Period

Frege begins “Function and Concept” by investigating the notion of a function as it originally appeared in higher analysis. Frege claims that the first place that a novel scientific expression receives a clear meaning is where it is required for the statement of a law. And this is the case with respect to functional expressions as they occur in analysis because there was a need to state general laws regarding functions. Throughout the essay Frege will clarify and extend the original meaning of a function until he has arrived at his mature conception of a concept that is a function from objects to truth-values. It is worth noting that Frege explicitly describes the content of his essay as supplementing the logic of the *Begriffsschrift* with new conceptions “whose necessity has occurred to [Frege] since”. What is new is the account of concepts and concept-extensions and they are necessary for the project of uncovering the covertly logical subject matter of arithmetic.

In the context in which the notion of a function was first introduced, functional expressions were constructed out of the arithmetical operations such as addition, multiplication, exponentiation, etc. and the arguments and values of these functions were taken to be the rational numbers. As mathematics developed, however, the notion of a function was extended along both these dimensions—the class of signs used to construct functional expressions was extended as was the range of things that served as their arguments and values. Thus in addition to the arithmetical operations, various means of transition to a limit were used to construct functional expressions, and the domain and range of functions were extended to include real numbers. Frege, confronted by a situation in which the original notion of a function has been extended to meet the demands of mathematical research, felt licensed to himself extend the notion of a function even further given the demands of his logical research. Moreover his extension fit the pattern in which the notion of a function had been extended within mathematical practice:

- Frege allowed novel signs to be the constituents of functional expressions, and

- Frege extended the class of things that could be the arguments and values of functions.

Frege's first innovation is to extend the class of functional expressions to include predicate expressions such as the identity sign and the greater than sign familiar from arithmetic. Thus not only does:

$$2x^3 + x$$

count as a functional expression, but now so does:

$$x^2 = 1$$

Frege observes that just as we can substitute the numeral "3" in for "2" in the expression:

$$2 * 2^3 + 2$$

to arrive at the well-formed expression:

$$2 * 3^3 + 3$$

so can we substitute "Nero" in for "Caesar" in the sentence

> Caesar conquered Gaul

to arrive at the well-formed expression:

> Nero conquered Gaul

That part of the complex expression that remained invariant under the substitution of number words is a functional expression as originally conceived. Frege's suggestion in the *Begiffsschrift* was that that part of the complex expression that remained invariant under the substitution of "Nero" for "Caesar" was similarly a functional expression albeit broadly conceived. Notice that number words and ordinary proper names aren't themselves functional expressions but are, in Frege's mature vocabulary complete expressions as are the original complex expressions and the ones arrived at by means of the substitutions. We can understand Frege's syntactic proposal as something like the following principle:

> Let $A[\ldots e \ldots]$ be a complex complete expression whether a singular term or a sentence and let $e$ be a complete expression that is a grammatical constituent of $A[\ldots e \ldots]$. Let $e'$ be a complete expression that is of the same grammatical category as $e$. That part of the complex expression that remains invariant under the substitution of $e'$ for $e$ in $A[\ldots e \ldots]$ is a functional expression, $A[\ldots (\quad) \ldots]$.

Having extended the class of functional expressions to include predicates, Frege immediately faces the question, what are the values of these novel functional expressions? Just as syntactic analogies guided Frege's decision to extend the class of functional expressions to include predicates, semantic analogies between functional expressions as traditionally conceived and predicate expressions guide Frege in his decision as to the values of these novel functional expressions. Consider the result of substituting "3" in for "2" in the complex expression "$2 * 2^3 + 2$":

$$2 * 3^3 + 3$$

Since the number words aren't codesignative—they differ in denotation—the denotation of the complex expressions also differ (given the function denoted by that part of the complex expression that remained invariant under the substitution). Notice that just as the number words "3" and "2" aren't codesignative neither are the proper names, "Nero" and "Caesar". And just as the denotation shifts as a result of the substation in the arithmetical example, so the truth-value shifts as the result of substituting "Nero" in for "Caesar" in the sentence:

Caesar conquered Gaul.

While the original sentence is true, the sentence resulting from the substitution:

Nero conquered Gaul

is false. This semantic analogy suggests that truth-values are the values of these novel functional expressions. We thus arrive at one of the clarifications that we have sought—just as predicates form a subclass of the class of functional expressions, the denotations of predicates, that is, concepts, form a subclass of the class of functions broadly conceived. In particular, concepts are functions from objects to truth-values.

Let's pause to consider two commitments of Frege's identification of truth-values as the values of a concept understood as special kind of function:

- If truth-values are the values of concepts, then truth-values are the denotations of sentences
- Since the values of functions are saturated entities, that is, objects, truth-values are themselves special kinds of objects

Recall Frege has just suggested that just as "$2^2$" denotes 4, "$2^2 = 1$" denotes the False. Similarly, all of the following sentences all denote the same thing, namely the True:

$$2^2 = 4$$
$$2 > 1$$
$$2 * 4 = 4 * 2$$

The problem is this: despite the fact that each of these sentences denote the same thing, the True, intuitively they seem to say different things and hence express different thoughts. So how can they really be codesignative?

The problem can be put in the form of a puzzle, an inconsistent set of claims:

1. The thought expressed by a sentence is its denotation.
2. The denotation of a sentence is its truth-value.
3. The sentences “$2^2 = 4$” and “$2 > 1$” each denote the True.
4. The sentences “$2^2 = 4$” and “$2 > 1$” express different thoughts.

If the thought expressed by a sentence is its denotation, and the denotation of a sentence is its truth-value, then the thought expressed by a sentence simply consists in the truth-value it denotes. It follows that since the sentences “$2^2 = 4$” and “$2 > 1$” each denote the True, they each express the same thought. This however conflicts with the final claim that the sentences “$2^2 = 4$” and “$2 > 1$” express different thoughts. We have a contradiction and so claims 1–4 cannot be rationally maintained together. We must reject at least one of these claims on pain of inconsistency. Call this *The Puzzle about Truth-Value*: How can sentences with the same truth-value differ in cognitive value.

The objection that Frege is presently considering consists in the suggestion that the best way out of this puzzle is to reject the second claim, that the denotation of a sentence is its truth-value. Frege, in effect, responds by suggesting that the inconsistency is just as easily avoided by rejecting the first claim, the claim that the denotation of a sentence is the thought expressed. From a formal perspective, each are equally good ways of avoiding the charge of inconsistency. So why should we prefer Frege’s way out of this puzzle over the objectors’?

Frege goes on to suggest that there is independent reason to distinguish the denotation of a sentence from the thought expressed. The independent reason is provided by the following argument:

1. *Compositionality of denotation*. Let $e$ and $e'$ be expressions belonging to the same syntactic category, and let $S[\ldots e' \ldots]$ be the result of substituting $e'$ in for at least one occurrence of $e$ in $S[\ldots e \ldots]$. $S[\ldots e \ldots]$ and $S[\ldots e' \ldots]$ denote the same thing just in case $e$ and $e'$ denote the same thing.

2. The names "Morning Star" and "Evening Star" denote the same thing, the planet Venus.

3. The sentences:

   - The Morning Star is a planet with a shorter period of revolution than the Earth. (a)
   - The Evening Star is a planet with a shorter period of revolution than the Earth. (b)

   differ only in the substitution of codesignative terms and so denote the same thing (from 1 and 2).

4. If it is possible to regard one sentence $S$ as true while regarding another sentence $S'$ as false then $S$ and $S'$ express different thoughts.

5. It is possible to regard (a) as true and (b) as false.

6. (a) and (b) express different thoughts. (from 4 and 5)

7. Since (a) and (b) denote the same thing and yet express different thoughts, the denotation of a sentence cannot be the thought expressed.

Notice, Frege's argument does not rely on the controversial claim that the denotation of a sentence is its truth-value. Let the denotation of a sentence be some as of yet determined value. Frege argues that given the compositionality principle, and given the principle that a difference in cognitive value make for a difference in the thought expressed, we have reason to distinguish the denotation of a sentence from the thought expressed—a reason that does not rely on truth values being the values of concepts and hence the denotations of sentences. Notice as well, while the argument is formally similar to the canonical argument associated with The Puzzle About Substitution, its target is different. The present argument seeks to establish that the the thought expressed by a sentence is distinct from its denotation. The argument associated with Frege's eponymous puzzle seeks to establish that the meaning of a name is distinct from its denotation.

In order to complete his logicist program Frege needed to specify the *values* of the functions that concepts are identified with, and, it is Frege's identification of truth-values as the values of concepts that leads him to draw the distinction between sense and denotation.

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# §5 "On Sense and Denotation" Revisited

Let's return to the discussion of truth in "On Sense and Denotation". That discussion and the prominent role that it plays (taking up, as it does, the bulk of that essay) was puzzling since one can state The Puzzle About Substitution without the controversial assumption that the denotation of a sentence is its truth-value. If, however, Frege was not primarily concerned with The Puzzle About Substitution but with The Puzzle About Truth Value, then our puzzle is resolved. Moreover, as we have seen, there is a direct connection between the unfinished business of the *Foundations* and The Puzzle About Truth Value.

Recall, Frege defends four distinguishable theses:

1. Sentences have denotations
2. The denotations of sentences are truth values
3. Names and sentences belong to the same logical category—in Frege's mature vocabulary, they are each *complete* expressions
4. Truth values are objects.

In defending these, Frege is extending and elaborating his claim, put forward in "Function and Concept", that the values of the functions with which concepts are idnetified are truth values. Let's review these in turn.

## §5.1 Sentences Have Denotations

Why believe that sentences have denotations? Frege argues as follows:

> The fact that we concern ourselves at all about the *Beduetung* of a part of a sentence indicates that we generally recognize and expect a *Bedeutung* for the sentence itself. The thought loses value for us as soon as we recognize that the *Bedeutung* of one of its parts is missing.

So, we would not concern ourselves with the denotation of words if we weren't interested in the denotations of the sentences of which they are a part. But we do, in fact, concern ourselves with the denotations of parts of sentences—we very often care whether an expression (say "the greatest prime number") denotes something.

That our interest in the denotation of words depends on and derives from our interest in the denotation of sentences is, perhaps, an expression or refinement of Frege's *context principle*:

> never to ask the meaning of a word in isolation, but only in the context of a proposition

Kant claimed that the fundamental unit of cognition or understanding is judgment: "all acts of understanding can be reduced to judgments, the understanding may be defined as the faculty of judgment." Subject and predicate terms are understood solely in terms of their role in judgment. A concept for Kant is a predicate of a possible judgment which is why "the only use which the understanding can make of concepts is to form judgments by them." One always begins with the content of a possible judgment, and anything else has a content only insofar as it contributes to the content of a possible judgment. Frege, following Kant, emphasizes the primacy of judgment. Reviewing his life's work, Frege writes:

> I start out from judgments and their contents, and not from concepts … instead of putting a judgment together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of a possible judgment.

Not only is the primacy of judgment playing a role in this argument, but so is Frege's compositionality principle, though with Frege's bifurcation of meaning comes two compositionality principles, the compositionality of sense and the compositionality of denotation. Specifically, then, Frege's argument makes implicit use of the compositionality of denotation—we are interested in the denotation of words only insofar as we are interested in the denotations of sentences, in part, because the latter is functionally dependent upon the former.

### §5.2 The Denotations of Sentences Are Truth-Values

Frege's argument so far might provide evidence that sentences have denotations, but it remains silent on what these denotations might be.

Might the thought expressed by a sentence be its denotation? By the compositionality of denotation, one can substitute codesignative words in a sentence without altering the denotation of a sentence. However, it is possible to substitute codesignative words in a sentence and alter the thought expressed. And so the thought expressed by a sentence is not its denotation. (Notice that this is the same argument that Frege gives in "Function and Concept". Notice as well that it is an *argument* that makes explicit the role of compositionality as a premise—this contrasts with the inaugural discussion of identity sentences where the argument for the sense–denotation distinction is at best gestured at.)

What then might the denotation of a sentence be? Frege twice describes his answer—that the denotation of a sentence is its truth-value—as a conjecture and the bulk of "On Sense and Denotation" is concerned with various "tests" of that conjecture. This signals that his argument is nondeductive in form. Nevertheless, it is an interesting argument in that it deepens his case for sentences having denotations. We are interested in the denotations of words only insofar as we are interested in the denotations of sentences because the denotations of sentences are truth-values and the truth-values of sentences have normative implications for our use of them in assertion and reasoning:

> But now why do we want every proper name to have not only a sense, but a Bedeutung? Why is the thought not enough for us? Because, and to that extent, we are concerned with its truth-value. ... It is the striving for truth that drives us always to advance from the sense to the Bedeutung.

### §5.3 Names and Sentences Belong to the Same Logical Category

That names and sentences belong to the same logical category is not so much argued for in "On Sense and Denotation" as it is presupposed. Names and sentences belong to the same logical category insofar as both are *complete* expressions and so contrast with *incomplete* or functional expressions. There is no question of *defining* the class of complete expressions, the notion of a complete expression is logically primitive and so not susceptible to definition. However, Frege maintained that there are range of analogies between names and sentences that reveal their unity. Some of these occur in "On Sense and Denotation". Here are two that play a prominent role in Frege's thought:

1. To determine the denotation of a complex sentence, all you need to know is the denotations of the constituent sentences and their method of compounding; similarly, to determine the denotation of a complex name all you need to know is the denotation of the constituent names and their method of compounding.

2. You can exchange truths for truths (or falsehoods for falsehoods) without changing the truth-value of the complex sentence; similarly, you can exchange names without affecting the denotation of the complex name as long as they are codesignative.

There are others, but none of them are terribly compelling. The real argument for this claim, if it is one, is that it is a consequence of Frege's functional analysis of predicates. Not only does his grammatical characterization of functional expressions presuppose that names and sentences belong to the unitary class of complete expressions, but if the denotation of a name is the argument of the function denoted by a predicate which determines the denotation of the sentence, then names and sentences must, to that extent, be on a par—they each consitute the doman and range of the functions with which concepts are identified.

### §5.4 Truth-Values Are Objects

Frege's thesis that truth-values are objects is not meant to be a substantive peice of metaphysics so much as a, perhaps surprising, consequence of his logical theory. The denotations of complete expressions are *saturated entities*, and the denotations incomplete expressions are *unsaturated entities*. Notice the order of explanation. Something is saturated or unsaturated just insofar as they are the denotations of complete or incomplete expressions. Frege does not try to explain these logical categories in terms of the kind of entities that expressions belonging to these categories denote. These logical categories are not explained by the relevant metaphysics so much as the relevant metaphysics is a reflection or reification of these logical categories. Thus having established that sentences are complete expressions, the thesis that truth-values are objects is secured—objects just are saturated entities, the denotations of complete expressions.

### §5.5 The Relation Between Our Puzzles

Frege's thesis that names and sentences belong to the same logical category has an important consequence for how The Puzzle About Substitution and The Puzzle About Truth Value are related. Modulo this controversial claim of Fregean grammar, these puzzles are equivalent.

Every instance of The Puzzle About Substitution is an instance of The Puzzle About Truth Value. Let $S$ and $S'$ differ only in the substitution of codesignative names and yet differ in cognitive value. Since the substitution of codesignative names always preserves truth value, $S$ and $S'$ also have the same truth-value and yet differ in cognitive value.

What about the other direction: Every instance of The Puzzle About Truth Value is an instance of The Puzzle About Substitution. Let $S$ and $S'$ have the same truth-value and yet differ in cognitive value. Sentences are names of truth values. Thus $S$ and $S'$ are codesignative names. $S$ and $S'$ differ in the substitution of codesignative names (trivially) and yet they differ in cognitive value.

## §6 Conclusion